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Structure of Malicious Singularities

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In this paper, we investigate relativistic spacetimes, together with their singular boundaries (including the strongest singularities of the Big Bang type, called malicious singularities), as noncommutative spaces. Such a space is defined by a noncommutative algebra on the transformation groupoid $\Gamma = \bar{E} \times G$, where $\bar{E}$ is the total space of the frame bundle over spacetime with its singular boundary, and G is its structural group. We show that there exists the bijective correspondence between unitary representations of the groupoid Γ and the systems of imprimitivity of the group G. This allows us to apply the Mackey theorem to this case, and deduce from it some information concerning singular fibers of the groupoid Γ. At regular points the group representation, which is a part of the corresponding system of imprimitivity, does not have discrete components, whereas at the malicious singularity such a group representation can be a single representation (in particular, an irreducible one) or a direct sum of such representations. A subgroup $K \subset G$, from which—according to the Mackey theorem—the representation is induced to the whole of G, can be regarded as measuring the "richness" of the singularity structure. In this sense, the structure of malicious singularities is richer than those of milder ones.

KEY WORDS: Singularity; singular boundary; groupoid; system of imprimitivity.

1. INTRODUCTION

Among various kinds of singularities one meets in studying solutions to Einstein field equations there are some of especially difficult character. They deserved names such as strong curvature singularities (Rudnicki *et al.*, 2002; Tipler, 1977a,b), crushing singularities (Eardley and Smarr, 1979), and malicious singularities (Heller and Sasin, 2002). These classes do not necessarily coincide, but

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there are singularities that belong to all of them. Typical examples are the initial and final singularities in Roberston–Walker–Friedman–Lemaître (RWFL) cosmological models and the central singularity in the Schwarzschild solution. They are usually described as elements (ideal points) of various singular boundaries of spacetime, but one hardly knows anything about their geometric nature besides their defining properties, i.e., that the spacetime curvature blows up as one approaches such a singularity, that it compresses any suitabley defined volume to zero, or that the fiber in the frame bundle over the singularity degenerates to a single point. In the present paper, we make an attempt to say something more about the malicious singularity by looking directly into its geometric nature.

To do so we must leave the category of smooth manifolds within which singularities can be reached only by a kind of a limiting process. In this paper, we treat spacetime, together with its singular boundary, as a noncommutative space. Although such a space is, in principle, nonlocal (the concepts of point and its neighborhood are meaningless in it), it can be studied in terms of representations of a certain noncommutative algebra in Hilbert spaces. It turns out that such an approach can provide some information about the nature of singularities. This method has been elaborated in the framework of our program of studying singularities (Heller and Sasin, 1994, 1996, 1999, 2002). The present paper is a direct continuation of the work of Heller and Sasin (2002); here we continue to focus on malicious singularities and, by consequently applying to them the theory of representations, we try to obtain a deeper insight into their geometric and physical nature.

The geometric context of our approach is the following: First, we consider singularities as elements of the b-boundary of spacetime (Schmidt, 1971). Although serious difficulties arise when this construction is applied to spacetime with some strong singularities, such as the ones in the closed RWFL model or in the Schwarzschild solution (Bosshard, 1976; Johnson, 1977), they can be overcome if the b-boundary construction is carried out in the category of structured spaces (Heller and Sasin, 2002). Schmidt's construction of the b-boundary of spacetime consists in defining the Cauchy completion $\bar{E}$ of the total space of the frame bundle E over spacetime M (with the help of a Riemann metric on E), and "projecting it down" (by the action of the Lorentz group that is a structured group of the frame bundle) to obtain the b-completed spacetime $\bar{M} = M \cup \partial_b M$, where $\partial_b M$ is the b-boundary of spacetime M. An element of $\partial_b M$ is said to be a *malicious singularity* if the fiber in $\bar{E}$ over it consists of a single point. Second, we construct the transformation groupoid $\Gamma = \bar{E} \times G$, where G is the structral group of the frame bundle, and a suitable noncommutative algebra $\mathcal{A}$ on it. This algebra plays the analogous role to the algebra of smooth functions on a manifold and defines a noncommutative space that encodes "noncommutative properties" of spacetime with singularities. It is a standard method of changing the usual space into a noncommutative space (Connes, 1994, pp. 99–103). And finally, we use the groupoid representation theory (or a representation of a suitable algebra on this groupoid) to investigate the structure of both singular and nonsingular groupoid fibers. Happily enough, there exists the bijective correspondence between unitary representations of the transformation groupoid $\Gamma = \bar{E} \times G$ and the systems of imprimitivity of the Lie group G. This fact allows us to reduce the problem to the study of unitary representations of the group G in a Hilbert space (which are better known than groupoid representations).

Our main result is that at any regular (nonsingular) point of spacetime the unitary representation, being a part of the system of imprimitivity of the group G, does not contain discrete components (G has no discrete series of representations). This is not true as far as malicious singularities are concerned. In this case, the condition for the system of imprimitivity is satisfied trivially. In particular, the corresponding group representation can be a single irreducible representation or a direct sum of such representations.

Spacetimes with b-boundaries are truly malicious geometric objects. This is demonstrated, among others, by the fact that the initial and final singularities in the closed RWFL cosmological model form a single "point" in the corresponding b-boundary, and are not Hausdorff separated from the rest of spacetime (Bosshard, 1976; Johnson, 1977). Our analysis does not change these conclusions, but the geometric tools used by us are powerful enough to give us some insight into such seemingly untractable situation. It is no longer a pathology, but rather a mathematical structure that could be used, if necessary, to model physical reality.

The organization of our material is the following: In Section 2, we present some elements of the groupoid structure. Our notation is basically the same as in Heller and Sasin (2002), but to make the present paper self-consistent we repeat some definitions and prepare necessary tools from the theory of group representations and systems of imprimitivity. In Section 3, we establish the bijective correspondence between representations of the transformation groupoid $\Gamma = \bar{E} \times G$ and the system of imprimitivity of the group G. Our main results, concerning both regular points and malicious singularities, are obtained in Section 4, and are illustrated in a typical example of the two-dimensional closed RWFL world model in Section 5. Finally, in Section 6, we collect some comments and interpretative remarks.

2. MATHEMATICAL PRELIMINARIES

2.1. Groupoids and Their Representations

We begin with a brief description of the groupoid concept (see, for instance, Paterson, 1999, Ch. 1) mainly to fix notation. *Groupoid* is a set Γ with a distinguished subset $\Gamma^2 \subset \Gamma \times \Gamma$, called the *set of composable elements*, together with two mappings:

$\cdot : \Gamma^2 \to \Gamma$ defined by $(x, y) \mapsto x \cdot y$, called *multiplication*, and
${}^{-1} : \Gamma \to \Gamma$ defined by $x \mapsto x^{-1}$ such that $(x^{-1})^{-1} = x$, called inversion.

Both mappings are supposed to satisfy the following conditions:

(i) if $(x, y), (y, z) \in \Gamma^2$ then $(xy, z), (x, yz) \in \Gamma^2$ and $(xy)z = x(yz)$,
(ii) $(y, y^{-1}) \in \Gamma^2$ for all $y \in \Gamma$, and if $(x, y) \in \Gamma^2$ then $x^{-1}(xy) = y$ and $(xy)y^{-1} = x$.

We also define the *set of units* $\Gamma^0 = \{xx^{-1}: x \in \Gamma\} \subset \Gamma$, and the following mappings:

$d : \Gamma \to \Gamma^0$ by $d(x) = x^{-1}x$, called *source mapping*, and
$r : \Gamma \to \Gamma^0$ by $r(x) = xx^{-1}$, called *target mapping*.

Let us notice that $(x, y) \in \Gamma^2$ if and only if $d(x) = r(y)$.
For each $u \in \Gamma^0$ we define the sets

$$\Gamma_u = \{x \in \Gamma: d(x) = u\} = d^{-1}(u)$$

and

$$\Gamma^u = \{x \in \Gamma: r(x) = u\} = r^{-1}(u).$$

Both these sets give different fibrations of Γ. The set $\Gamma_u^u := \Gamma^u \cap \Gamma_u$ is closed under multiplication and inverse. It is called the *isotropy group* at u.

The above construction is purely algebraic, but it can be equipped with the smoothness structure. In this case, it is called a *smooth* or *Lie groupoid* (Paterson, 1999, Ch. 2.3).

The so-called transformation groupoids (or action groupoids) form an important class of Lie groupoids. Let E be a differential manifold with a group G acting on it to the right, $E \times G \to E$. This action leads to the bundle $(E, \pi_M, M = E/G)$. The Cartesian product $\Gamma = E \times G$ has the structure of a groupoid, and is called a *transformation groupoid*. The elements of Γ are pairs $\gamma = (p, g)$, where $p \in E$ and $g \in G$. Two such pairs $\gamma_1 = (p, g)$ and $\gamma_2 = (pg, h)$ are composed in the following way:

$$\gamma_2\gamma_1 = (pg, h)(p, g) = (p, gh).$$

The inverse of (p, g) is (pg, g^{-1}). We could think on $\gamma = (p, g)$ as an arrow beginning at p and ending at pg. Two arrows γ_1 and γ_2 can be composed if the beginning of γ_2 coincides with the end of γ_1.

The set of units is

$$\Gamma^0 = \{\gamma^{-1}\gamma: \gamma \in \Gamma\} = \{(p, e): p \in E\}.$$

We shall often consider the “fibers” of this groupoid:

$$\Gamma_{(p,e)} = \{(p, g): g \in G\},$$
$$\Gamma^{(p,e)} = \{(ph^{-1}, h): h \in G\}.$$

In the following, we shall abbreviate the symbols $\Gamma_{(p,e)}$ and $\Gamma^{(p,e)}$ to Γ_p and Γ^p, respectively. If an element $\gamma = (p, g) \in \Gamma$ is represented as an arrow from p to pg, the set Γ_p should be thought of as the set of arrows which begin in (p, e), and the set Γ^p as the set of arrows which end at (p, e).

In what follows, we shall assume that G is a unimodular group (the Haar measure exists on G). Since all fibers of the groupoid Γ are isomorphic with G, the Haar system can be defined on Γ, and Γ can be regarded as a locally compact Hausdorff groupoid (Paterson, 1999, p. 32).

Let us now recall the definition of a groupoid representation. Let Γ be a locally compact groupoid, Γ^0 its space of units, and $(\Gamma^0, \{H_u\}_{u\in\Gamma^0}, \mu)$ a Hilbert bundle. Here $\{H_u\}$ is a collection of Hilbert spaces with u ranging over Γ^0, and μ is a probability measure on Γ^0. By a *section* of the Hilbert bundle we mean a function $f : \Gamma^0 \to \bigcup_{u\in\Gamma^0} H_u$, where $f(u) \in H_u$.

Definition 2.1. A *representation* $\mathcal{U}$ of the locally compact groupoid Γ is given by a Hilbert bundle $(\Gamma^0, \{H_u\}_{u\in\Gamma^0}, \mu)$, where μ is a quasi-invariant measure on Γ^0, and a mapping $\Gamma \ni x \to L(x) \in B(H_{d(x)}, H_{r(x)})$, where d and r are the source and the range mappings, respectively. L is supposed to satisfy the following conditions:

(i) $L(u) = \mathrm{id}_{H_u}$, $u \in \Gamma^0$,
(ii) $L(x)L(y) = L(x \cdot y)$, almost everywhere with respect to the groupoid measure, for all $x, y \in \Gamma$ that can be composed with each other (in the case considered in the present paper this condition is satisfied everywhere),
(iii) $L(x)^{-1} = L(x^{-1})$, almost everywhere, for every $x \in G$,
(iv) for any two sections $\xi, \eta \in (L^2(\Gamma^0, \{H_u\}_{u\in\Gamma^0}, \mu)$ of the Hilbert bundle, the function

$$x \to (L(x)\xi(d(x)), \eta(r(x)))$$

is measurable on Γ (Landsman, 1998, pp. 282–285).

2.2. Induced Representations and Systems of Imprimitivity

Let G be a unimodular Lie group and K its closed subgroup (therefore, K is also unimodular). In such a case, there exists on $M = K\backslash G$ (the set of right cosets) a G-invariant measure. Let further (L, V) be a unitary representation of the group K in a Hilbert space V (which can also be finitely dimensional). Now, we form the Hilbert space $H_L = L^2(M, V, d\mu)$ of a new representation of the group G. H_L consists of functions defined on G with values in V such that

(i) the function $g \to (f(g), v)$, for every $g \in G$ and $v \in V$, is measurable (with respect to the Haar measure dg on G),

(ii) $f(kg) = L(k) f(g)$ for every $k \in K$ and $g \in G$ (covariance condition),

(iii) $\int_M \| f(g)\|^2 \, d[g] < \infty$, where $[g] = Kg$.

The space H_L with the scalar product

$$(f \mid f')_{H_L} = \int_M (f(g),\ f'(g))_v \, d[g]$$

is indeed the Hilbert space.

Let us define the operator

$$U^L(g_0) f(g) = f(gg_0).$$

Definition 2.2. The representation (U^L, H_L) of the group G is called the *induced representation* of G from the subgroup K through the representation (L, V).

This representation is unitary with respect to the above scalar product (this follows from the invariance of the measure). Let us notice that the regular representation[4] $(R, L^2(G))$ of the group G is the induced representation from the trivial subgroup $\{e\}$ by (L, V), with $L = 1$ and $V = \mathbf{C}$.

Let again G be a unimodular Lie group, (U, H) its unitary representation in a Hilbert space H, and M a G-space, i.e., a space with a (right) action of G (the action is not necessarily transitive). Let further P be a spectral measure on M, i.e., a measure on Borel subsets of M with values in the space of projection operators in the Hilbert space H. If $B \subset M$ is a Borel subset then $P(B)$ is an orthogonal projection in H.

Definition 2.3. A quadruple (G, U, M, P) is a *system of imprimitivity* (S.I. for short) of the group G for the representation U with the base M if the following conditions are satisfied:

(i) $P(M) = \mathrm{id}_H$,

(ii) $U(g)P(B)U(g^{-1}) = P(Bg^{-1})$

for every $g \in G$ and $B \subset M$, B being a Borel set.

Condition (ii) expresses a "covariance" of P with respect to U. S.I. is said to be *transitive* if G acts transitively on M. In such a case, $M = K\backslash G$, where K is a closed subgroup of G.

There exists another (equivalent) definition of S.I.

[4] Let us recall that the *left regular representation* of a unimodular Lie group G in the Hilbert space $L^2(G, dg)$ is given by $U(g) = L_g$, where $L_g f(x) := f(g^{-1}x)$, and the *right regular representation* of G by $U(g) = R_g$, where $U(g) f(x) = f(xg)$, for every $g \in G$.

Definition 2.4. S.I. of the group G for the representation U with the base M is the quadruple (G, U, M, π), where (π, H) is a nondegenerate representation of the $*$-algebra $C_0(M)$ of continuous functions on M vanishing at infinity in a Hilbert space H. Conditions (i) and (ii) from the previous definition are now replaced by

$$U(g)\pi(f)U(g^{-1}) = \pi(R_g f),$$

where $R_g f(x) = f(xg)$, $x \in M$, $g \in G$, $f \in C_0(M)$. S.I. defined in this way is said to be *smooth* if (π, H) is a nondegenerate representation of the algebra $C_0^\infty(M)$ of smooth functions on M vanishing at infinity.

Theorem 2.5. (*Mackey, theorem*). *If (G, U, M, P) is a transitive S.I. (i.e., $M = K\backslash G$) then the representation (U, H) of the group G is induced from its subgroup K, or more precisely there exists a unitary representation (L, U_L) of the subgroup $K \subset G$ and the isomorphism of Hilbert spaces $J : H \to H_L$ such that*

$$JU(g)J^{-1} = U^L(g),$$
$$JP(B)J^{-1} = P^L(B)$$

for every $g \in G$ and every Borel subset $B \subset M$. In other words the representations U and U^L are unitary equivalent (*Mackey, 1952*). □

3. SYSTEMS OF IMPRIMITIVITY AND REPRESENTATIONS OF THE TRANSFORMATION GROUPOID

In this section, we find the correspondence between representations of the transformation groupoid $\Gamma = E \times G$ and systems of imprimitivity of the group G. It is given by the following theorem.

Theorem 3.1. *Let (G, U, X, π) be the S.I. of the group G for the representation U with base X, and let $\mathcal{U}$ be a representation of the transformation groupoid $\Gamma = X \times G$. There exists a one-to-one correspondence*

$$\{(G, U, X, \pi)\} \leftrightarrow \{\mathcal{U}\}.$$

Proof: The proof is a combination of Theorem 3.4.4, Corollary 3.4.6, and Corollary 3.7.4 from Landsman (1998), and Theorem 3.1.1 from Paterson (1999) [see also formula (3.20) from the book by Paterson]. □

We shall now directly construct the above correspondence for the differential groupoid $\Gamma = \overline{E} \times G$.

Step 1. In this step we will construct another realization of S.I. for our case. We choose a point $p_0 \in E$ such that $\tau(p_0) = m$, where $\tau : E \to M$ is the canonical projection, and construct the space

$$\mathcal{F}_G(E_m, H) = \left\{\psi : E_m \to H\colon \psi(p_0 g) = U(g^{-1})\psi(p_0)\right\},$$

where H is a Hilbert space of the representation U (or of π). The space $\mathcal{F}_G$ consists of continuous functions.[5] We equip this space with the scalar product

$$(\psi_1|\psi_2) = (\psi_1(p_0), \psi_2(p_0))_H,$$

changing it into a Hilbert space. We define the operator $\bar{U}$ on the space $\mathcal{F}_G$

$$[\bar{U}(g)\psi](p) = U(g)\psi(p),$$

and the representation $\bar{\pi}$ of $C_0(E)$ in the space $\mathcal{F}_G$

$$[\bar{\pi}(f)\psi](p_0 g) = \pi(R_{g^{-1}} f)\psi(p_0 g)$$

for every $\psi \in \mathcal{F}_G(E_m, H)$, $f \in C_0(E)$, and for a point p_0 such that $\tau(p_0) = m$; that is to say

$$[\bar{\pi}(f)\psi](p_0) = \pi(f)\psi(p_0).$$

This condition enforces $(G, \bar{U}, E, \bar{\pi})$ to be an S.I.

Proposition 3.2. *$(G, \bar{U}, E, \bar{\pi})$ is an S.I. of the group G for the representation $\bar{U}$ with base E.*

Proof: By using the covariance of the S.I. (G, U, E, π) and the properties of functions form $\mathcal{F}_G$ we check the condition

$$\psi \in \mathcal{F}_G \Rightarrow \bar{\pi}(f) \in \mathcal{F}_G$$

and the convariance condition for $(G, \bar{U}, E, \bar{\pi})$. □

Step 2. First, we construct a Hilbert space which will from the Hilbert bundle. Let $p_0 \in E$, and $p_1 = p_0 g_0$. Then

$$\mathcal{H}^{p_0} = \left\{F : \Gamma^{p0} \to H\colon F\left(p_0 g^{-1}, g\right) = U(g)F(p_0, e)\right\}.$$

Of course, functions F are continuous, and we have the Hilbert bundle $(E, \mathcal{H}^p, d\mu)$, where $d\mu$ is the measure on E.

Now, we define the representation operator of the groupoid $\Gamma = E \times G$

$$\mathcal{U}(p_0, g_0) : \mathcal{H}^{p_0} \to \mathcal{H}^{p_1}$$

[5] Strong continuity is assumed as a part of the definition of the unitary representation of a Lie group; i.e., it is assumed that, for every $h \in H$, the function $G \in g \to U(g)h \in H$ is continuous.

by

$$[\mathcal{U}(p_0, g_0)F] = F(\gamma^{-1}\eta) = F\left(p_1 g^{-1}, g g_0^{-1}\right).$$

Here $\gamma = (p_0, g_0)$, $\eta = (p_1 g^{-1}, g)$.

Unitarity of the operator $\mathcal{U}(p_0, g_0)$ is implied by the definitions of the scalar products in H^{p_0} and H^{p_1}:

$$\begin{aligned}(F_1, F_2)_{\mathcal{H}^{p_0}} &= (F_1(p_0, e), F_2(p_0, e))_H,\\ (\bar{F}_1, \bar{F}_2)_{\mathcal{H}^{p_1}} &= (\bar{F}_1(p_1, e), \bar{F}_2(p_1, e))_H.\end{aligned}$$

We can easily check that all conditions of the groupoid representation are satisfied (in this case, “almost everywhere” is replaced by “everywhere”).

Step 3. Now, we should check that the constructed groupoid representation corresponds to the initial S.I. To this end, let us define the isomorphism of Hilbert spaces

$$J_{p_0} : \mathcal{H}^{p_0} \to \mathcal{F}_G(E_m, H),$$

where $m = \tau(p_0)$, by

$$\psi\left(p_0 g^{-1}\right) = F\left(p_0 g^{-1}, g\right),$$

where $J_{p_0}F = \psi$.

Theorem 3.3. *The isomorphisms J_p “transform operators $\mathcal{U}(p_0, g_0)$ onto operators $\bar{U}(g_0^{-1})$” in the sense that $\mathcal{U}(p_0, g_0) = J_{p_1}^{-1} \circ \bar{U}(g^{-1}) \circ J_{p_0}$. In other words, the following diagram commutes:*

$$\begin{tikzcd} \mathcal{H}^{p_0} \arrow[r, "\mathcal{J}_{p_0}"] \arrow[d, "\mathcal{U}(p_0, g_0)"'] & \mathcal{F}_G(E_m, H) \arrow[d, "\bar{U}(g_0^{-1})"] \\ \mathcal{H}^{p_1} \arrow[r, "\mathcal{J}_{p_1}"'] & \mathcal{F}_G(E_m, H) \end{tikzcd}$$

Proof: The proof is by direct computation. □

4. SYSTEMS OF IMPRIMITIVITY FOR SINGULAR SPACETIMES

Let us notice that the groupoid Γ is the disjoint sum of $\Gamma_m = E_m \times G$, i.e., $\Gamma = \bigcup_{m \in M} \Gamma_m$. And if the malicious singularity is present at m_1, $\Gamma = \bigcup_{m \in M} \Gamma_m \cup \Gamma_{m_1}$, where $\Gamma_{m_1} = \{(0, 0, \ldots, 0)\} \times G$.

Definition 4.1. Let $m_0 \in \bar{M}$, and $\bar{M} = M \cup \{m_1\}$. The *local* S.I. at the point m_0 of the group $G = SO(3, 1)$ for the representation (U, H) of G is $(G, U, \underline{E}_{m_0}, \pi)$. Let us notice that the base of this S.I. is E_{m_0}.

Proposition 4.2. *Let $m \in M$ be a regular point. The S.I. (G, U, E, π) determines the local S.I. at the point m: (G, U, E_m, π_1).*

Proof: Let us consider the algebra $C_0(E_m)$ of continuous functions on E_m vanishing at infinity. We choose a point $p_0 \in E_m$, and want to show that $f \in C_0(E_m)$ can be "extended" to $\tilde{f} \in C_0(E)$.

Let $\{(\mathcal{O}_n, f_n)\}_{n \in \mathbf{N}}$ be the approximate unit for the algebra $C_0(M)$; $\mathcal{O}_n$ is here a sequence of sets such that the closure $\bar{\mathcal{O}}_n$ of each of them is compact, and $\operatorname{supp} f_n \subset \mathcal{O}_n$. We also assume that every $\mathcal{O}_n$ is the domain of trivialization of the bundle $E \to M$. Let further

$$\tilde{f}_n(m, g) = f_n(m) \cdot f(p_0 g).$$

Of course, $\tilde{f}_n \in C_0(E)$. Finally, we define the representation π_1 of the algebra $C_0(E_m)$ in the space H:

$$\pi_1(f) = \lim_{n \to \infty} \pi(\tilde{f}_n)$$

where the limit is understood in the sense of strong topology on the Hilbert space H. □

Theorem 4.3. *Let $(G, \bar{U}, E_m, \bar{\pi})$ be a local S.I. at a regular point $m \in M$. Then the representation $(\bar{U}, \mathcal{F}_G(E_m, H))$, and consequently the representation (U, H), is unitary equivalent to the factor representation of the regular representation of the group G in the Hilbert space $L^2(G)$.*

Proof: Let us notice that $E_m = K \backslash G$, where $K = \{e\}$. Therefore, the considered S.I. is transitive. On the strength of the Mackey theorem, the representation (U, H) is equivalent to the induced representation from the subgroup $K = \{e\}$. The inducing representation is given by the operator $L = \mathrm{id}_v$. (If the subgroup is trivial, the only representation operator is the multiplication by 1, but the representation space V can be n-dimensional.) Consequently, the representation induced by L is given by the factor representation containing the regular representation in the space $L^2(G)$, with the multiplicity equal to $\dim(V)$. □

Corollary 4.4. *The representation (U, H), being a part of the local S.I., does not contain discrete irreducible components.*

Proof: The regular representation of the group $G = SO(3, 1)$ has no discrete series. □

Let us notice, however, that this result depends on the dimension of space. The group $SO(n, 1)$ has no discrete series for $n = 2k + 1$, but it has the discrete series for $n = 2k$.

Let us now consider the situation in the maliciously singular fiber; such a fiber is $\Gamma_{m_1} = \{pt\} \times G$, where $\tau(pt) = m_1 \in \bar{M} \backslash M$. In fact, Γ_{m_1} can be regarded as a well-defined groupoid (indeed, $(pt, g_1) \circ (pt, g_2) = (pt, g_2 g_1)$), and we can consider the space $\mathcal{H}^{pt}$. If $F \in \mathcal{H}^{pt}$ then

$$F(pt, g) = U(g)F(pt, e).$$

We see that the operator $U(g)$ acts according to the rule, but in the trivial way. The same is true for the operator of the groupoid representation

$$[\mathcal{U}(pt, g_0)F](pt, g) = F\left(pt, gg_0^{-1}\right) = U\left(g_0^{-1}\right)F(pt, g).$$

Let now (G, U, E_{m_1}, π) be the local S.I. at the point pt. We have $C_0(E_{m_1}) \simeq \mathbf{R}$, and the condition of imprimitivity

$$U(g)\pi(f)U(g^{-1}) = \pi(f), \qquad f = \text{const}, \qquad \pi(f) = a\, \mathrm{id}_H$$

is satisfied trivially.

This means that if (G, U, E_{m_1}, π) is the local S.I. at the maliciously singular point pt, then the condition for S.I. does not impose any limitations on the representation (U, H). In particular, it can be an irreducible representation. □

5. EXAMPLE: TWO-DIMENSIONAL RWFL WORLD MODEL

In this section, we consider a simplified (two-dimensional) RWFL cosmological model with its two malicious singularities that often serves as a typical example in the classical singularity problem (Bosshard, 1976; Dodson, 1978).

Let us consider the spacetime

$$M = \{(\eta, \chi) \colon \eta \in (0, T), \chi \in S^1\},$$

where $(O, T) \subset \mathbf{R}$, carrying the metric

$$ds^2 = R^2(\eta)(-d\eta^2 + d\chi^2).$$

This model has the initial singularity: $R^2(\eta) \to 0$ as $\eta \to 0$, and the final singularity: $R^2(\eta) \to 0$ as $\eta \to T$ (for the detailed presentation of this model see Dodson, 1978, or Heller and Sasin, 2002).

To make a contact with our previous construction, let us list all relevant magnitudes:

$$M = (0, T) \times S^1, \qquad (\eta, \chi, \lambda) \in E, \quad t \in \mathbf{R},$$
$$\gamma = (\eta, \chi, \lambda, t) \in \Gamma, \qquad d(\gamma) = (\eta, \chi, \lambda),$$
$$r(\gamma) = (\eta, \chi, \lambda + t), \qquad \Gamma^P = \{(pt^{-1}, t): t \in \mathcal{R}\} = \{(\eta, \chi, \lambda - t, t\}.$$

To obtain the groupoid representation corresponding to a given representation (U, H) of the group $G \simeq \mathcal{R}$, we construct the Hilbert space for a chosen regular point $p_0 = (\eta, \chi, t_0)$:

$$\mathcal{H} = \left\{F : \Gamma^{p_0} \to H : F\left(p_0 g^{-1}, g\right) = U(g)F(p_0, e)\right\}$$
$$= \{F(\eta, \chi, \lambda_0 - t, t) = U(t)F(\eta, \chi, \lambda_0, 0)\}.$$

And for the groupoid representation operator we have

$$\mathcal{U}(p_0, g_0)F := \mathcal{U}(\eta, \chi, \lambda_0, t_0)F(\eta, \chi, \lambda_0 + t_0 - t, t)$$
$$= F(\eta, \chi, \lambda_0 + t_0 - t, t - t_0) = U(-t_0)F(\eta, \chi, \lambda_0 - t, t).$$

To obtain the corresponding S.I. $(G, U, E_{(\eta,\chi)}, P)$, for $G = \mathbf{R}$, we make use of the generalized Stone, Neimark, Ambrose, Godement theorem (see Barut and Rączka, 1977, p. 160), which says that a representation (U, H) of the group $\mathbf{R}$ in any Hilbert space can be expressed with the help of a spectral measure P, in the following way:

$$U(t) = \int_{\boldsymbol{R}} e^{its}\, dP(s).$$

We have

$$\mathcal{F}_G(E_m, H) = \{\psi : E_m \to H : \psi(\eta, \chi, \lambda_0 + t) = U(t)\psi(\eta, \chi, \lambda_0)\}.$$

In this Hilbert space the spectral measure is

$$P(B)\psi(\eta, \chi, t) = \chi_B(t)\psi(\eta, \chi, t),$$

where $B \subset \mathbf{R}$ is a Borel set, and χ_B its characteristic function.

It can be easily seen that the system (G, U, E_m, P) indeed satisfies conditions of S.I. Therefore, the results obtained in the previous sections remain valid. For regular points, the representation (U, H) is equivalent to the regular representation of the group $\mathbf{R}$ in $L^2(\mathbf{R})$, possibly with the multiplicity greater than 1. For malicious singularities, every representation (U, H) of the group $\mathbf{R}$ satisfied the conditions of S.I. The regular representation of $\mathbf{R}$ in $L^2(\mathbf{R})$ has, exactly as for $SO(3, 1)$, no discrete components.

6. INTERPRETATION AND COMMENTS

So far our results were purely formal; let us now try to read from them a physical meaning. In physical applications systems of imprimitivity appear in the following circumstances.

Let us consider a quantum physical system having the symmetry group P. It is described by a pair $(U(P), H)$, where $U(P)$ is a unitary representation of the group P in a Hilbert space H. Let us further assume that a classical system is described by the pair (P, M), where M is the space of a classical observable that characterizes the state of this system (e.g. the space of positions or space of momenta), and P is acting on M as its symmetry group. If in H there is a state ψ_x in which the value of an observable is $x \in M$, we say that the quantum state ψ_x corresponds to the classical magnitude x. Let us denote

$$H_x = \{\psi_\alpha \in H : a = x\}.$$

If such correspondence exists, i.e., if the quantum system has an interpretation in terms of classical observables, the following conditions hold:

(i) $H_x = \bigcup_{x \in M} H_x$,
(ii) $U(p)H_x \subset H_{px}$,

and there exists the system of imprimitivity for the representation $U(p)$ of the symmetry group P (Mensky, 1976). The above is visualized in the following diagram, the left column of which represents quantum description and its right column the corresponding classical description.

$$\begin{tikzcd}
H_x \ni \psi_x \arrow[rr, leftrightarrow] \arrow[d, "p \in P"'] & & x \in M \arrow[d, "p \in P"] \\
U(p)\psi_x \arrow[rr, leftrightarrow] & & px \in M
\end{tikzcd}$$

If P acts on M transitively, i.e., if there is a subgroup $K \subset P$ such that $M = K \backslash P$, then, on the strength of the Mackey theorem, any imprimitive representation of the group P is induced from the subgroup K (Mackey, 1978, 1998).

Let us now apply this analysis to the case of spacetime with malicious singularities. The groupoid representation is given by the pair $(\mathcal{U}, \{H_u\}_{u \in E})$. Although in the present work we consider classical singularities, we can say that the above pair provides a quantum description of the singularity (or something analogous to quantum description since it uses typically quantum mathematical tools). We also have its classical description given by the action of the group $G = SO(3, 1)$ on E, $E \times G \to E$ (Lorentz rotations of local frames). Since the groupoid representation $\mathcal{U}$ corresponds bijectively to the system of imprimitivity (G, U, E, π), we could say that the quantum description of our model corresponds to its classical description. This some-how justifies the fact that although we are facing the classical singularity problem, it can be dealt with in terms of mathematical structures typical for quantum theory (unitary operators, Hilbert spaces, etc.).

In our case, the Mackey theorem says that the unitary representation of the Lorentz group $G = SO(3, 1)$, which is the part of the corresponding S.I., is induced from its subgroup K such that $E_m = K\backslash G$. If $m \in M$ is a regular point then $K = \{e\}$; if $m \in \bar{M}\backslash M$ is a malicious singularity then $K = G$. This means that in our model the correspondence between quantum description and classical description is complete if we do not take into account malicious singularities. At maliciously singular points this correspondence formally also takes place, but the S.I. condition is always trivially satisfied.

There can exist "intermediate" singularities for which the isotropy group K is a proper subgroup of G; they are not regular points of spacetime, but as singularities are weaker than malicious ones (e.g., see Ellis and Schmidt, 1977). Let K_p be the isotropy group of a point $p \in E$. We have $K_p = K_q$ if there is $g \in G$ such that $q = pg$, and $E_m = K_p\backslash G$. Since E_m, for an "intermediate" singularity at m, is a quotient space, the Mackey theorem applies, and consequently the represenation, that is a part of the S.I. with the base E_m, is an induced representation by a certain representation of the subgroup K.

Let us notice that if m is a regular point of spacetime, $\dim E_m = \dim G$; if m is an "intermediate" singularity, $\dim E_m = \dim G - \dim K$; if m is a malicious singularity, $\dim E_m = 0$. In this sense, K may be regarded as measuring the "strength" of a given singularity.

At regular points the group representation, which is an element of S.I., does not have discrete components [the group $SO(3, 1)$ has no discrete series]. In the quantum field theory this implies the impossibility to localize an elementary particle. At the malicious singularity such a group representation can be a single irreducible representation or a direct sum of such representations. Formally speaking, this would mean that at the singularity elementary particles can be localized. Since, however, this follows from the fact that the S.I. condition does not impose any limitations on what can happen here, the correct interpretation seems to be that general relativity is essentially an incomplete theory: malicious singularities are its "open windows" that claim for a more general (and more complete) theory. This is not true, however, that we know nothing about the nature of the malicious singularity; as we have shown, some of its characteristics surrender to the analysis in terms of representations in Hilbert spaces.

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